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THE APPLICATION OF THE SEPARATION PRINCIPLE FOR THE LINEAR CONTINUOUS SYSTEMS WITH COLOURED NOISE

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Abstract. This paper shows that the separation principle is applicable for the optimal control problem of linear continuous stochastic systems with not only white but also coloured noise. We shall give the explicit form of the optimal control in the scalar autonomous case.

1. INTRODUCTION

The Wonham's separation principle gives the optimal control for the system

$$dx = Axdt + Budt + GdW\,, \tag{1.1}$$

$$dy = Hxdt + RdV\,, \tag{1.2}$$

$$J(u) = E\left(\int_0^T L(t, x(t)\,,\ u(t))dt + \Psi[x(T)]\right)\,, \tag{1.3}$$

where y is the observation, u is the control and $J(u)$ is the expected performance index. [3. pp 188–195] The processes $x = x(t)$, $y = y(t)$, $u = u(t)$ are random vector processes and A, B, G, H, R are matrices which usually depend on time t. $L(t, x, u)$ is a nonnegative, $\psi(x)$ is an arbitrary real function. $W = W(t)$, $V = V(t)$ are independent brownian motions adapted to the same increasing family of σ -algebra, $\{F_t\}$.

The question is raised whether the optimal control can be given in the case of other kind of observations. We show that the separation principle is applicable for determining the optimal control when $x(t)$ satisfies (1.1) and the observation satisfies the following system:

$$y = Cx + z\,, \tag{1.4}$$

$$dz = Nzdt + RdV\,, \tag{1.5}$$

where C, N, R are real matrices, and z is coloured noise defined by (1.5). First using the Kalman–Bucy filter model we get the mean square optimal estimate of $x, \widehat{x}(t)$ then we control the system corresponding to $\widehat{x}(t)$. The proof is essentially different from that of the original separation principle, because applying Ito's formula we get such stochastic differential equation that depends not only on x, but on y and W too. In practice it is a natural condition that the additive noise is cloured noise. Balakrishnan dealt with similar problems in other approach. [5].

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2. THE APPLICATION OF THE SEPARATION PRINCIPLE IN THE CASE OF COLOURED NOISE

Consider the system of equations, (1.1), (1.4), (1.5). Using Ito's formula transform the observation y into a stochastic differential equation:

$$\begin{aligned} dy &= \dot{C}xdt + Cdx + dz = \\ &= \dot{C}xdt + C[Axdt + Budt + GdW] + N[y - Cx]dt + RdV\,, \end{aligned} \tag{2.1}$$

where we get the matrix, $\dot{C}$ by differentiating with respect to time, t the entries of C. Introducing $H\ := \dot{C} + CA - NC$ we have the system equivalent to (1.1), (1.4), (1.5):

$$\begin{aligned} dx &= Axdt + Budt + GdW\,, \\ dy &= Hxdt + Nydt + CBudt + CGdW + RdV\,. \end{aligned} \tag{2.2}$$

Hence we assume that each matrix of (1.1), (2.2) is bounded in $[0,T]$. Consider the following two systems:

$$dx_0 = Ax_0dt + GdW, \qquad x_0(0) = x(0), \tag{2.3}$$

$$dy_0 = Hx_0dt + CGdW + RdV, \qquad y_0(0) = 0; \tag{2.4}$$

$$dx_+ = Ax_+dt + Budt, \qquad x_+(0) = 0, \tag{2.5}$$

$$dy_+ = Hx_+dt + Nydt + CBudt, \qquad y_+(0) = y(0), \tag{2.6}$$

where y is the observation in (2.2). Notice that

$$x = x_0 + x_+\,, \tag{2.7}$$

$$y = y_0 + y_+\,. \tag{2.8}$$

We use the following notations:

$$G_t^0 = \sigma(y_0(r), \quad r \le t)\,, \tag{2.9}$$

$$G_t^u = \sigma(y(r),\ r \le t)\,, \tag{2.10}$$

$$\widehat{x}_0(t) = E(x_0(t) \mid G_t^0)\,, \tag{2.11}$$

$$\widehat{x}(t) = E(x(t) \mid G_t^u)\,, \tag{2.12}$$

where $\sigma(\cdot)$ denotes the generated σ –algebra, and $E(\cdot|\cdot)$ denotes the conditional expected value.

Lemma 2.1

If $u(t)$ is G_t^0–measurable and $G_t^0 = G_t^u$, then:
a) The conditional distribution of $x(t)$ gives G_t^u , $P(x(t) \mid G_t^u)$ is gaussian with mean

$$\widehat{x}(t) = \widehat{x}_0(t) + x_+(t)\,, \tag{2.13}$$

and the covariance matrix of $x(t) - \widehat{x}(t)$ is the symmetrical, positive definite solution of the following Riccati–type matrix differential equation:

$$\dot{P} = AP + PA' - [GG'C' + PH'](CGG'C' + RR')^{-1}[GG'C' + PH']' + GG'\,, \tag{2.14}$$

$$P(0) = cov x(0)\,, \tag{2.15}$$

where the transpose of G is G', and *cov* $x(0)$ denotes the covariance matrix of $x(0)$.

b) The conditional mean $\widehat{x}(t)$ obey the stochatic differential equation:

$$d\widehat{x} = [A\widehat{x} + Bu]dt + F[dy_0 - H\widehat{x}_0 dt]\,, \tag{2.16}$$

$$\widehat{x}(0) = Ex(0)\,, \tag{2.17}$$

where

$$F = [GG'C' + PH'][CGG'C' + RR']^{-1}\,, \tag{2.18}$$

and

$$dy_0 - H\widehat{x}_0 dt = (B \circ B)^{\frac{1}{2}} d\widetilde{W}\,, \tag{2.19}$$

where $\widetilde{W}$ is brownian motion and

$$(B \circ B) = [CGG'C' + RR']\,. \tag{2.20}$$

PROOF. Taking the conditional mean of both sides of (2.7) we get

$$\widehat{x}(t) = \widehat{x}_0(t) + E(x_+(t) \mid G_t^u)\,, \tag{2.21}$$

thus for (2.13) it is sufficient to show that $x_+(t)$ is G_t^u –measurable. Using the well–known solving formula of linear differential equations we get the explicite form of $x_+(t)$ from (2.5):

$$x_+(t) = \int_0^t \Phi(t)\Phi^{-1}(\sigma)B(\sigma)u(\sigma)d\sigma\,, \tag{2.22}$$

where $\Phi(t)$ is the matrix solution of the homogeneous differential equation:

$$\frac{dx}{dt} = A(t)x(t)\,. \tag{2.23}$$

This means that the measureability of $u(t)$ implies that of $x_+(t)$. Moreover (2.7), (2.13) imply $x - \widehat{x} = x_0 - \widehat{x}_0$ and $cov(x - \widehat{x}) = cov(x_0 - \widehat{x}_0)$. According to the Kalman–filtering [2. pp 425.] $cov(x - \widehat{x}_0)$ obeys the differential equation (2.14) with the initial condition (2.15) and $\widehat{x}_0$ satisfies the stochastic differential equation:

$$d\widehat{x}_0 = A\widehat{x}_0 dt + F[dy_0 - H\widehat{x}_0 dt]\,, \tag{2.24}$$

$$\widehat{x}_0(0) = Ex(0)\,. \tag{2.25}$$

Finally adding (2.5) to (2.24), and using (2.13) we get (2.16) and (2.17). □
Remark that on conditions of lemma 2.1. $y_+(t)$ is also $G_t^u = G_t^0$ –measurable.
Make the following transformation:

$$dy_0 - H\widehat{x}_0 dt = d(y - y_+) - H(\widehat{x} - x_+)dt = dy - (H\widehat{x} + Ny + CBu)dt\,. \tag{2.26}$$

From (2.16), (2.19) we obtain

$$d\widehat{x} = [A\widehat{x} + Bu]dt + F[dy - (H\widehat{x} + Ny + CBu)dt]\,, \tag{2.27}$$

and

$$d\widehat{x} = [A\widehat{x} + Bu]dt + F[B \circ B)^{\frac{1}{2}} d\widetilde{W}\,. \tag{2.28}$$

Now we associate the new system (2.28), (2.17) with such new performance $\widehat{L}$, $\widehat{\Psi}$ that the expected performance of the new system be equal to the expected performance of the original system for the control $u(t)$ in lemma 2.1.
Let

$$\widehat{L}(t, \widehat{x}, u) = \int_{R^n} L(t, x, u) g(t, x - \widehat{x}) dx\,, \tag{2.29}$$

$$\widehat{\Psi}(\widehat{x}) = \int_{R^n} \Psi(x) g(T, x - \widehat{x}) dx\,, \tag{2.30}$$

$$\widehat{J}(u) = E\left(\int_0^T \widehat{L}(t, \widehat{x}(t), u(t))dt + \widehat{\Psi}[\widehat{x}(T)] \right)\,, \tag{2.31}$$

where

$$g(t,x) = (2\pi)^{-\frac{n}{2}} (\det P(t))^{-\frac{1}{2}} \exp\left[-\frac{1}{2} x' P(t)^{-1} x \right]\,, \tag{2.32}$$

and the matrix $P(t)$ is the symmetrical, positive definite solution of (2.14), (2.15) , the number n is the dimension of the vector $x(t)$, and we denote the determinant of $P(t)$ by $\det P(t)$.

Lemma 2.2

If $u(t)$ is G_t^0 -measurable and $G_t^0 = G_t^u$ then

$$J(u) = \widehat{J}(u).$$

PROOF. By definition of $\widehat{L}$ and G_t^0 -measurability of $u(t)$

$$E(L(t, x(t), u(t)) \mid G_t^0) = \widehat{L}(t, \widehat{x}(t), u(t)) \tag{2.33}$$

with probability 1. Hence

$$E\int_0^T L(t, x(t), u(t)dt = E\int_0^T E(L|G_t^0)dt = E\int_0^T \widehat{L}dt, \tag{2.34}$$

and similarly

$$E\Psi(x(T)) = E\widehat{\Psi}(\widehat{x}(T)). \quad \Box \tag{2.35}$$

Lemma 2.3

Let C^k denote the space of continuous functions from $[0, T]$ into $\mathbf{R}^k$ with the sup norm $\| \;\|$. Let Φ be an operator in C^k such that, for some constant K,

$$\| \Phi g - \Phi h \| \leq K \| g - h \|, \quad \text{all} \quad g, h \in C^k.$$

Then, given $h(t) \in C^k$, the equation

$$g(t) = h(t) + \int_0^t (\Phi g)(r)dr \tag{2.36}$$

has a unique solution $g \in C^k$. The sequence of successive approximations $g_0 \equiv 0$,

$$g_{\ell+1}(t) = h(t) + \int_0^t (\Phi g_\ell)(r)dr, \quad (\ell = 0, 1, 2, \dots) \tag{2.37}$$

converges in norm to g.

The proof can be found in [3] pp. 191. $\Box$
Let U denote the finite dimensional control set.
Let Γ denote the space of functions γ which satisfy:

$a)\, \gamma : [0, T] \times C^k \to U$ is Borel measurable. (2.38)

$b)\, \gamma(t, 0)$is bounded. (2.39)

$c)$ For each $\gamma \in \Gamma$ there exists a constant K_γ such that (2.40)

$\| \gamma(t, g) - \gamma(t, h) \|_u \leq K_\gamma \| g - h \|$ for all $g, h \in C^k$and $\quad 0 \leq t \leq T$,
where $\| \cdot \|_u$ is the Euclidean norm over U.

$d)$ $If g(r) = h$ for $0 \leq r \leq t$, then $\gamma(t, g) = \gamma(t, h)$. (2.41)

Lemma 2.4

If $u(t) = \gamma(t, y)$ for some γ of Γ, where $y = y(t)$ is the observation in (1.4.), and $y(0)$ is G_t^0 -measurable, then

$$a)\; u(t) \text{ is } G_t^u\text{–measurable,} \tag{2.42}$$

$$b) G_t^u = G_t^0. \tag{2.43}$$

PROOF. (2.10) implies that $y(r)$ G_t^u -measurable if $r \geq t$.

Let

$$y_t(r) = \begin{cases} y(r), & r \le t\,, \\ y(t), & r > t\,. \end{cases} \tag{2.44}$$

then $y_t(r)$ G_t^u -measurable. From (2.38), (2.41)

$$u(t) = \gamma(t, y) = \gamma(t, y_t)\,, \tag{2.45}$$

so $u(t)$ is G_t^u -measurable.
To prove $G_t^u = G_t^0$ we must first show that

$$G_t^0 \subset G_t^u\,. \tag{2.46}$$

Let Φ be the fundamental solution of equation (2.23). By (2.22) and (2.6) we get

$$\begin{aligned} y_+(t) = y(0) + \int_0^t H(\tau)\left[\int_0^\tau \Phi(\tau)\Phi^{-1}(\sigma)B(\sigma)u(\sigma)d\sigma\right] + \\ + N(\tau)y(\tau) + C(\tau)u(\tau)d\tau\,. \end{aligned} \tag{2.47}$$

If $\sigma, \tau \le t$, then $u(\sigma)$ and $y(\tau)$ are G_t^u-measurable, and by (2.47) so is $y_+(t)$. By (2.8) $y_0(r)$ is also G_t^u -measurable if $r \le t$. From (2.9) we get (2.46). To prove $G_t^u \subset G_t^0$ we define an operator $\xi : \mathcal{C}^k \to \mathcal{C}^k$:

$$\begin{aligned} (\xi g)(t) = H(t)\int_0^t \Phi(t)\Phi^{-1}(\sigma)B(\sigma)\gamma(\sigma, g)d\sigma + \\ + C(t)B(t)\gamma(t, g) + N(t)g(t)\,. \end{aligned} \tag{2.48}$$

By the estimation (2.40) and the boundness condition for matrices of (2.48) ξ satisfies the hypothesis of lemma 2.3. If follows from (2.8), (2.47) that $y(t)$ is the solution of integral equation

$$y(t) = y_0(t) + y(0) + \int_0^t (\xi y)(\tau)d\tau\,. \tag{2.49}$$

Define the following iteration:

$$\begin{aligned} & y^{(0)} \equiv 0 \\ & y^{(\ell+1)}(t) = y_0(t) + y(0) + \int_0^t (\xi y^{(\ell)})(\tau)d\tau\,, \quad (\ell = 0, 1, 2, \dots)\,. \end{aligned} \tag{2.50}$$

Then by lemma 2.3. $\| y^{(\ell)}(t) - y(t) \| \to 0$ $(\ell \to \infty)$. Since $y(0)$ is G_t^0-measurable we get recursively that $y^{(\ell)}(t)$ is also F_t^0-measurable. As the distributions are gaussian we get that $y(t)$ is also G_t^0-measurable, which means by (2.10) that $G_t^u \subset G_t^0$ holds. □

Let $\mathcal{A}$ denote the space of controls that satisfy:

$$u : [0, T] \to U, \tag{2.51}$$

$$u(t) = \gamma(t, y) \text{ for some } \gamma \text{ of } \Gamma, \tag{2.52}$$

$$E\int_0^T |u(t)|^k dt < \infty \text{, for all } k = 0, 1, 2, \dots\,, \tag{2.53}$$

$u(t)$ is adapted to $\{F_t\}$, (2.54)

there exists the strong solution of the system, (2.2). (2.55)

The elements of set $\mathcal{A}$ are called feasible controls. The minimizing problem correspondig to the system (2.2) is

$$J(u) \to \min_{\mathcal{A}}\,, \tag{2.56}$$

where $J(u)$ is defined by (1.3).
Note that the G_t^0-measurability of $y(0)$ implies that for all feasible controls the hypothesis of lemma 2.2 are satisfied that is $J(u) = \widehat{J}(u)$.

Now we explain the idea of the separation principle. The partially observed stochastic control problem is converted into a completely observed problem, namely the estimate $\widehat{x}(t)$ satisfies (2.17), (2.28) which is taken as a new system with a new expected performance $\widehat{J}(u)$. Let $\mathcal{B}$ denote the set of controls that satisfy (2.51), (2.53), (2.54), then the minimizing problem corresponding to the completely observed problem is

$$\widehat{J}(u) \to \min_{\mathcal{B}} . \tag{2.57}$$

Since $\mathcal{A} \subset \mathcal{B}$, if the minimum in (2.57) is reached for a control $u(t)$ belonging to $\mathcal{A}$ then this $u(t)$ is the optimal control in (2.56) as well.

We show that it is the case when L, Ψ are quadratic. The well-known fact that the explicite form of optimal control for the completely observed system can be given when L, Ψ are quadratic will be used. [4. pp. 221.]
Let C_1, D_1, F_1 be symmetrical, positive matrices and

$$J(u) = E\left(\int_0^T [x'(r)C_1(r)x(r) + u(r)'D_1(r)u(r)]dr + x(T)'F_1(T)x(T)\right) , \tag{2.58}$$

where $x = x(r)$ is the solution of the partially observed system (2.2) corresponding to the control $u(t)$, and x' is the transpose of vector x.
Next we determine $\widehat{J}(u)$.
Using the notation $\widetilde{x} = x - \widehat{x}$ we have

$$\begin{aligned} \widehat{L}(t,\widehat{x},u) &= \int_{R^n} (x'C_1x + u'D_1u)g(t, x - \widehat{x})dx = \\ &= \int_{R^n} (\widetilde{x} + \widehat{x})'C_1(\widetilde{x} + \widehat{x})g(t,\widetilde{x})d\widetilde{x} + u'D_1u . \end{aligned} \tag{2.59}$$

From the identities

$$1 = \int_{R^n} g(t,\widetilde{x})d\widetilde{x} , \tag{2.60}$$

$$0 = \int_{R^n} \widetilde{x}g(t,\widetilde{x})d\widetilde{x} , \tag{2.61}$$

and

$$(\widetilde{x} + \widehat{x})'C_1(\widetilde{x} - \widehat{x}) = \widetilde{x}'C_1\widetilde{x} + \widetilde{x}'C_1\widehat{x} + \widehat{x}'C_1\widetilde{x} + \widehat{x}'C_1\widehat{x} \tag{2.62}$$

we get

$$\widehat{L}(t,\widehat{x},u) + \widehat{x}'C_1\widehat{x} + u'D_1u + \int_{R^n} \widetilde{x}'C_1\widetilde{x}g(t,\widetilde{x})d\widetilde{x} , \tag{2.63}$$

and

$$\widehat{\Psi}(\widehat{x}) = \widehat{x}'F_1(T)\widehat{x} + \int_{R^n} \widetilde{x}'F_1(T)\widetilde{x}g(T,\widetilde{x})d\widetilde{x} . \tag{2.64}$$

Thus

$$\begin{aligned} \widehat{J}(u) = E(&\int_0^T (\widehat{x}(t)'C_1(t)\widehat{x}(t) + u'(t)D_1(t)u(t))dt + \\ &+ \widehat{x}(T)'F_1(T)\widehat{x}(T)) + Z , \end{aligned} \tag{2.65}$$

where

$$Z = E\left(\int_0^T \widetilde{x}(t)'C_1(t)\widetilde{x}(t)dt + \widetilde{x}(T)'F_1(T)\widetilde{x}(T)\right) . \tag{2.66}$$

From lemma 2.1. the estimation error $\tilde{x}(t)$ has a normal distribution with mean 0 and variance $P(t)$. It follows that Z is unaffected by the control.
Consider the following optimizing problem:

$$\widehat{J}(u) = E\left(\int_0^T (\widehat{x}'(t)C_1(t)\widehat{x}(t) + U'(t)D_1(t)u(t))dt + \widehat{x}(T)'F_1(T)\widehat{x}(T)\right), \tag{2.67}$$

$$\widehat{J}(u) \to \min_B . \tag{2.68}$$

Then the optimal control $u^*(t)$ for (2.17),. (2.28), (2.68) will be also optimal for (2.17), (2.28), (2.57). Moreover, as the performance functions in (2.67) are quadratic $u^*(t)$ has the following explicite form:

$$u^*(t) = G_1(t)\widehat{x}(t), \tag{2.69}$$

where

$$G_1(t) = -D_1^{-1}(t)B(t)'K(t), \tag{2.70}$$

and the matrix function $K(t)$ is the symmetrical, positive definite solution of the following Riccati-type matrix differential equation:

$$\dot{K} = -A'K - KA + KBD_1^{-1}B'K - C_1, \tag{2.71}$$

$$K(T) = F_1(T). \tag{2.72}$$

To prove that $u^*(t)$ is the optimal control for the original system (2.2), (2.56) we have to show that $u^*(t) \in \mathcal{A}$.
For this define γ_{G_1} as follows:

$$\gamma_{G_1}(t,y) = G_1(t)\widehat{x}(t), \tag{2.73}$$

where $y = y(t)$ is the observation, and $\widehat{x}(t)$ is the solution of the system (2.17), (2.28) when $u = u^*(t)$ is the optimal control given by (2.69) . Suppose that $G_1(t)$ has continuous derivate in $[0,T]$. To show $u^*(t) \in \mathcal{A}$ we have to prove that $\gamma_{G_1}(\cdot,\cdot) \in \Gamma$.
From (2.27) and (2.69) we get

$$d\widehat{x} = (A - FH + BG_1 - FCBG_1)\widehat{x}dt - FNydt + Fdy. \tag{2.74}$$

Let Φ_{G_1} denote the fundamental matrix solution of the homogeneous differential equation

$$\frac{dx}{dt} = (A - FH + BG_1 - FCBG_1)x. \tag{2.75}$$

It is well-known that the solution of the linear stochastic differential equation can be expressed in the following way [4. pp 146]

$$\widehat{x}(t) = \Phi_{G_1}(t)Ex(0) - \int_0^t \Phi_{G_1}(t)\Phi_{G_1}^{-1}(r)F(r)N(r)y(r)dr + \\ + \int_0^t \Phi_{G_1}(t)\Phi_{G_1}^{-1}(r)F(r)dy(r). \tag{2.76}$$

Thus we have the following explicite form of $\gamma_{G_1}(t,g)$:

$$\gamma_{G_1}(t,g) = G_1(t)(\Phi_{G_1}(t)Ex(0) - \int_0^t \Phi_{G_1}(t)\Phi_{G_1}^{-1}(r)F(r)N(r)g(r)dr + \\ + \int_0^t \Phi_{G_1}(t)\Phi_{G_1}^{-1}(r)F(r)dg(r)). \tag{2.77}$$

Integration by parts gives

$$\gamma_{G_1}(t,g) = G_1(t)(\Phi_{G_1}(t)Ex(0) - \int_0^t \Phi_{G_1}(t)\Phi_{G_1}^{-1}(r)F(r)N(r)g(r)dr + \\ + [\Phi_{G_1}(t)\Phi_{G_1}^{-1}(r)F(r)g(r)]_{r=0}^t - \int_0^t (\frac{\partial}{\partial r}[\Phi_{G_1}(t)\Phi_{G_1}^{-1}(r)F(r)])g(r)dr). \tag{2.78}$$

If we assume that all matrices appear in (2.2) are integrable in norm square on $[0,T]$ then it is easily seen that γ_{G_1} satisfies the conditions (2.38)–(2.41).
Therefore $\gamma_{G_1} \in \Gamma$ which implies $u^*(t) \in \mathcal{A}$. So we proved the following theorem.

Theorem 2.1

Consider the system

$$
\begin{aligned}
dx &= Axdt + Budt + GdW\,, \\
y &= Cx + z\,, \\
dz &= Nzdt + RdV\,, \\
J(u) &= E\left(\int_0^T [x'(r)C_1(r)x(r) + u(r)'D_1(r)u(r)]dr + x(T)'F_1(T)x(T)\right)\,, \\
J(u) &\to \min_{\mathcal{A}}\,,
\end{aligned}
$$

where V, W are independent brownian motions adapted to the same σ–algebras, $\{F_t\}$ and $\mathcal{A}$ is the space of the feasible controls (2.51)–(2.55).
Suppose, that A, B, C, C_1, D_1, F_1, G are integrable in norm square on $[0,T]$, $[CGG'C'+$ $+RR']^{-1}$ exists and is also integrable in norm square on $[0,T]$.
Assume that $y(0)$ is G_t^0–measurable, and G_1 in (2.70) has continuous derivate on $[0,T]$.
Then the optimal control $u^*(t)$ is the following

$$u^*(t) = G_1(t)\widehat{x}(t)\,,$$

where

$$
\begin{cases}
d\widehat{x} = (A - FH + BG_1 - FCBG_1)\widehat{x}dt - FNydt + Fdy\,, \\
\widehat{x}(0) = Ex(0)\,,
\end{cases}
$$

$H = \dot{C} + CA - NC\,,$
$F = [GG'C' + PH'][CGG' + RR']^{-1}\,,$
and P is the symmetrical, positive definite solution of the Riccati–type differential equation

$$
\begin{cases}
\dot{P} = AP + PA' - F[GG'C' + PH']' + GG'\,, \\
P(0) = cov\ x(0)\,,
\end{cases}
$$

and $G_1 = -D_1^{-1}B'K$, where K is the symmetrical, positive definite solution of the Riccati–type differential equation

$$
\begin{cases}
\dot{K} = -A'K - KA + KBD_1^{-1}B'K - C_1\,, \\
K(T) = F_1(T)\,.
\end{cases}
$$

3. THE SCALAR AUTONOMOUS STOCHASTIC CONTROL PROBLEM

When all matrices appear in theorem 2.1. are independent from time t the solutions of Riccati–type differential equations can be given in explicite form. [1. pp 82.] Thus we have the following theorem.

Theorem 3.1

Let A, B, C, C_1, D_1, F_1, G, N, R denote real numbers and $H = C(A - N)$.
Consider the system

$$
\begin{aligned}
dx &= Axdt + Bu + GdW\,, \\
y &= Cx + z\,, \\
dz &= Nzdt + RdV\,, \\
J(u) &= E\left(\int_0^T (C_1x^2(r) + D_1u^2(r))dr + F_1(T)x^2(T)\right)\,, \\
J(u) &\to \min_{\mathcal{A}}\,,
\end{aligned}
\tag{3.1}
$$

where V, W are independent 1–dimensional brownian motions adapted to the same σ–algebra, $\{F_t\}$ and $\mathcal{A}$ is the space of the feasible controls (2.51)–(2.55).
Suppose that $(CG)^2 + R^2 \neq 0$, and $y(0)$ is G_t^0–measurable. Then the optimal control $u^*(t)$ is the following

$$u^*(t) = -\frac{B}{D_1} K(t)\widehat{x}(t)\,, \tag{3.2}$$

where

$$K(t) = \frac{exp(2\widetilde{A}t)}{\widetilde{C}_0 - \frac{B^2(exp(2\widetilde{A}t)-1)}{2D_1\widetilde{A}}} + \widetilde{C}_1\,, \tag{3.3}$$

if $\widetilde{A} \neq 0$, and

$$K(t) = \frac{1}{\widetilde{C}_0 - \frac{B^2}{D_1}t} + \widetilde{C}_1 \tag{3.4}$$

if $\widetilde{A} = 0$.
The numbers $\widetilde{C}_0, \widetilde{C}_1, \widetilde{A}$ are defined by the equations

$$\frac{B^2}{D_1}\widetilde{C}_1^2 - 2A\widetilde{C}_1 - C_1 = 0\,, \tag{3.5}$$

$$\widetilde{A} = -A + \widetilde{C}_1 \frac{B^2}{D_1}\,, \tag{3.6}$$

$$\widetilde{C}_0 = \frac{exp(2\widetilde{A}T)}{F_1(T) - \widetilde{C}_1} + \frac{B^2(exp(2\widetilde{A}T) - 1)}{2D_1\widetilde{A}}\,, \tag{3.7}$$

if $\widetilde{A} \neq 0$ and

$$\widetilde{C}_0 = \frac{1}{F_1(T) - \widetilde{C}_1} + \frac{B^2}{D_1}T \tag{3.8}$$

if $\widetilde{A} = 0$.
Moreover $\widehat{x}(t)$ is the solution of the stochastic differential equation

$$\begin{cases} d\widehat{x} = (A - FH + BG_1 - FCBG_1)\widehat{x}dt - FNydt + Fdy\,, \\ \widehat{x}(0) = Ex(0)\,, \end{cases}$$

where

$$F(t) = \frac{G^2C + P(t)H}{(CG)^2 + R^2}\,, \tag{3.9}$$

and the function $P(t)$ is

$$P(t) = \frac{exp(2\widetilde{\widetilde{A}}t)}{\widetilde{\widetilde{C}}_0 + \frac{H^2(exp(2At)-1)}{2\widetilde{\widetilde{A}}((CG)^2+R^2)}} + \widetilde{\widetilde{C}}_1\,, \tag{3.10}$$

if $\widetilde{\widetilde{A}} \neq 0$, and

$$P(t) = \frac{1}{\widetilde{\widetilde{C}}_0 + \frac{H^2t}{(CG)^2+R^2}} + \widetilde{\widetilde{C}}_1\,, \tag{3.11}$$

if $\widetilde{\widetilde{A}} = 0$.
The real numbers $\widetilde{\widetilde{C}}_0, \widetilde{\widetilde{C}}_1, \widetilde{\widetilde{A}}$ are defined by the equations

$$-\frac{H^2}{(CG)^2 + R^2}\widetilde{\widetilde{C}}_1^2 + 2\left(A - \frac{G^2CH}{(CG)^2 + R^2}\right)\widetilde{\widetilde{C}}_1 + G^2 - \frac{(G^2C)^2}{(CG)^2 + R^2} = 0\,, \tag{3.12}$$

$$\widetilde{\widetilde{A}} = A - \frac{G^2CH}{(CG)^2 + R^2} - \frac{\widetilde{\widetilde{C}}_1 H^2}{(CG)^2 + R^2} \tag{3.13}$$

$$\widetilde{\widetilde{C}}_0 = -\frac{1}{\widetilde{\widetilde{C}}_1}\,. \tag{3.14}$$

Remark

By theorems 2.1, 3.1 we can easily write program for personal computers, which makes the model of the optimal control problem by discretizing method.

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